# Distributed Data-Driven Power Iteration for Strongly Connected Networks

Azwirman Gusrialdi and Zhihua Qu

***Abstract*— This paper presents data-driven power iteration to distributively estimate the dominant eigenvalues of an unknown linear time-invariant system. The proposed strategy only requires a single trajectory data or measurements. Furthermore, in order to perform the distributed estimation, the communication network topology can be chosen to be any strongly connected directed graphs. The proposed data-driven power iteration is demonstrated using several numerical examples and is then applied to estimate the generalized algebraic connectivity of cooperative systems and to control the epidemic spreading.**

***Index Terms*— Eigenvalue estimation, distributed algorithm, data-driven, power iteration, strongly connected network.**

## I. Introduction

The dominant (rightmost) eigenvalues of a linear (or linearized) time-invariant (LTI) system play an important role in the analysis and control of interconnected dynamical systems. For example, dominant eigenvalues are utilized in power system's small signal stability analysis to monitor inter-area oscillation [1]. Furthermore, the dominant eigenvalue also measures the convergence rate for reaching consensus among cooperative systems [2], [3].

Power iteration is an effective approach in estimating the largest (in terms of magnitude) eigenvalue together with the corresponding right eigenvector of a matrix. In combination with a deflation technique, it has been widely used to estimate the dominant eigenvalue in different application areas including multi-robot systems [2], power system [4], mechanical engineering systems [5], and epidemic spreading [6]. One of the key features of power iteration which makes it suitable for analyzing complex systems is its distributed implementation which follows naturally given that the matrix of interest has a sparse structure [7], [8].

In spite of its promising applications, knowledge on the matrix of interest or system model is necessary to perform the power iteration. However, the system model is often unknown or not available due to geographical constraint, privacy issue or simply because it is too complicated to obtain as observed in power system [9]. This motivates the development of data-driven approach to estimate the eigenvalues of a LTI system from available data/measurements and in the absence of model knowledge. However, to the best of our knowledge it is still not well-understood whether it is possible to adopt power iteration in order to estimate the dominant eigenvalues in the absence of the system model knowledge while preserving its distributed implementation.

The paper aims at answering the following question: can the power iteration be used to distributively estimate the dominant eigenvalues using available data and what is its limitation? In particular, we build our results based on the modified model-based power iteration presented in [8] which allows us to deal with complex dominant eigenvalues. Our main findings can be summarized as follow: (i) the proposed data-driven power iteration could estimate the dominant eigenvalues sequentially until the rightmost complex one. As one of our contributions we demonstrate that instead of the left eigenvector which is commonly used in the literature, the right eigenvector should be used in the deflation technique to enable data-driven estimation; (ii) the communication network topology can be chosen to be any strongly connected directed graphs independent from the physical interconnection topology. This provides an additional degree-of-freedom for choosing the communication network topology.

It is worth to note that there exists other alternative approaches to estimate eigenvalues of an LTI system using available data, see for example the work in [10]–[13]. However, those alternative methods suffer from at least one of the following limitations: (i) data from multiple experiments or trajectories are required to estimate the eigenvalues; (ii) the estimation is performed either in a centralized manner or the communication network topology is restricted to bidirectional network; (iii) the methods are not tailored to estimate only the dominant eigenvalues and thus extensive storages and communications are necessary to perform the estimation. It will be shown in the paper that the proposed data-driven power iteration can overcome all the above limitations.

The paper is organized as follows. Data-driven dominant eigenvalues estimation problem is formulated in Section II. In Section III, a summary of model-based power iteration is first provided followed by the proposed data-driven power iteration. The proposed data-driven power iteration is demonstrated via several numerical examples in Section IV. Concluding remarks and future work are presented in Section V.

## II. Problem statement

In this section, we first provide a brief overview of graph theory followed by the problem formulation.

### A. Preliminaries

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph (digraph) with a set of nodes $\mathcal{V} = \{1, 2, \cdots, n\}$ and a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. An edge $(i, j) \in \mathcal{E}$ denotes that node $i$ can obtain information from (or is influenced by) node $j$. The set of in-neighbors

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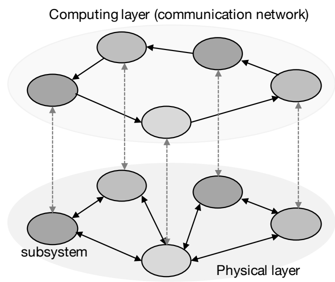


Fig. 1: Physically interconnected system with its communication network topology: the communication network topology is given by an arbitrary strongly connected digraph independent of the physical interconnection topology, i.e., structure of matrix $A$ in (1)

of node $i$ is denoted by $\mathcal{N}_i^{\text{in}} = \{j|(i,j) \in \mathcal{E}\}$. The directed graph $\mathcal{G}$ is *strongly connected* if every node can be reached from any other nodes by following a set of directed edges.

### B. Problem formulation

Consider an interconnected system of $q$ linear time invariant subsystems whose overall dynamics is described by

$$\dot{x} = Ax \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $x = [x_1^T, \cdots, x_q^T]^T$ and $x_i \in \mathbb{R}^{n_i}$ denotes the state of the $i$th subsystem with $n_1 + \cdots + n_q = n$. The eigenvalues of matrix $A$ are denoted by $\lambda_1(A), \cdots, \lambda_n(A)$ and ordered by decreasing real part, i.e. $i > j \Rightarrow \Re(\lambda_i(A)) \leq \Re(\lambda_j(A))$. It is assumed that all the eigenvalues $\lambda_i(A)$ are semi-simple, i.e., the eigenvectors of $A$ are linearly independent and the state $\|x(t)\|$ is bounded, that is $\Re(\lambda_1(A)) \leq 0$. It is worth noting that in practice the entries of matrix $A$ are unknown, for example due to privacy issue or simply because it is too complicated to obtain as can be observed in power systems [9], [10]. Therefore, in this paper it is assumed that matrix $A$ is unknown. Each subsystem has access to the number of subsystems $q$ and its own (noiseless) sampled state $x_i(k) \triangleq x_i(t)|_{t=kT}, (k = 0, 1, \cdots)$ with $T$ denotes the sampling time, corresponding to the discrete-time model of (1) given by

$$x(k+1) = A_d x(k) \tag{2}$$

where $A_d = e^{AT}$. Sampling time $T$ is chosen such that the above property of eigenvalues of $A$ is preserved for the eigenvalues of $A_d$. The subsystems can communicate and exchange information via a communication network whose topology is given by a strongly connected digraph $\mathcal{G}$ and its structure is independent of the structure of matrix $A$, see for example Fig. 1.

The objective of this paper is to estimate distributively the dominant (rightmost) eigenvalues of matrix $A$, that is eigenvalues located near the imaginary axis (with largest real-part), using only available single trajectory data $x_i(k)$ for $k = 0, 1, \cdots$. Power iteration in combination with deflation technique is a widely used method to distributively estimate the dominant eigenvalues of a matrix [14]. However, in general knowledge of system matrix is required to execute the power iteration. Therefore, in this paper we are interested in answering the following fundamental question: *Can the power iteration be used to distributively estimate the dominant eigenvalues of matrix $A$ using available data and what is its limitation?*

## III. Main result

The relationship between the eigenvalues of matrices $A$ and $A_d$ is given by

$$\lambda_i(A) = \frac{\ln(\lambda_i(A_d))}{T} \tag{3}$$

and dynamics (1) and (2) share the same eigenvectors. Since $i > j \Rightarrow \Re(\lambda_i(A)) \leq \Re(\lambda_j(A))$, eigenvalues of the discretized dynamics satisfy

$$|\lambda_1(A_d)| \geq |\lambda_2(A_d)| \geq \cdots \geq |\lambda_n(A_d)| \tag{4}$$

where $|\lambda_i(A_d)|$ denotes the magnitude of $\lambda_i(A_d)$. In this section, we first provide a summary of model-based power iteration followed by presenting the main result which is the data-driven power iteration.

### A. Model-based power iteration

If matrix $A_d$ in (2) is known or available, power iteration [14] can be readily applied to estimate $\lambda_1(A_d)$ and the eigenvalue of interest $\lambda_1(A)$ can then be computed from (3). However, the output of the standard power iteration is always a real number which prevents its application when $\lambda_1(A_d)$ is a complex number. To address this issue, the work [7], [8] proposed a modified power iteration which allows distributed computation of a complex eigenvalue. Here, the communication network topology $\mathcal{G}$ is set to be similar to the sparsity or structure of matrix $A_d$. In the following, we make a minor modification and summarize the distributed algorithm presented in [8], whose analysis and proof can be found in [8].

1) Each node performs the following iterations

$$z^{(1)}(k+1) = A_d z^{(1)}(k) \tag{5}$$

where $z^{(1)} = [(z_1^{(1)})^T, \cdots, (z_q^{(1)})^T]^T$. Suppose that $z^{(1)}(0)$ has a nonzero component in the direction of $w_1^{A_d}$, i.e., $(z^{(1)}(0))^T \cdot w_1^{A_d} \neq 0$ where $w_1^{A_d}$ denotes the right eigenvector corresponding to $\lambda_1(A_d)$.

2) Node $i$ stores the result of three consecutive iterations $z_\ell^{(1)}(k), z_\ell^{(1)}(k-1)$ and $z_\ell^{(1)}(k-2)$ with $z_\ell^{(1)} \in \mathbb{R}^{n_\ell}$ and $\ell \in i \cup \mathcal{N}_i^{\text{in}}$ where $\mathcal{N}_i^{\text{in}}$ denotes the in-neighbors of node $i$.

3) Each node solves for $b, d$ from the $\sum_{j\in\{i\cup\mathcal{N}_i^{\text{in}}\}} n_j$ linear equations

$$z_\ell^{(1)}(k) + bz_\ell^{(1)}(k-1) + dz_\ell^{(1)}(k-2) = 0$$

4) Each node computes the solutions $p_{i,1}(k)$ and $p_{i,2}(k)$ to the following quadratic equation

$$p_i^2 + bp_i + d = 0$$

5) Each node further computes

$$p_{i,3}(k) = \frac{z_{i,m}^{(1)}(k)}{z_{i,m}^{(1)}(k-1)}$$

for any $m = \{1, \cdots, n_i\}$ where $z_{i,m}^{(1)}$ denotes the $m$-th element of vector $z_i^{(1)}$

6) As $k \to \infty$, at least one of the estimates $p_{i,j}(k)$ converges to $\lambda_1(A_d)$ with $j = \{1, 2, 3\}$. Specifically,
   - if $\lambda_1(A_d)$ is complex, then $(p_{i,1}, p_{i,2})$ converge to $(\lambda_1(A_d), \lambda_2(A_d))$ where $\lambda_2(A_d)$ is the complex conjugate of $\lambda_1(A_d)$ while $p_{i,3}$ does not converge to any number
   - when $\lambda_1(A_d) \in \mathbb{R}$ and $\lambda_2(A_d)$ is complex, $p_{i,3}$ then converges to $\lambda_1(A_d)$ but $(p_{i,1}, p_{i,2})$ do not converge to any value
   - Finally, when $\lambda_1(A_d), \lambda_2(A_d) \in \mathbb{R}$, then $(p_{i,1}, p_{i,2})$ converge to $(\lambda_1(A_d), \lambda_2(A_d))$ or $(\lambda_2(A_d), \lambda_1(A_d))$ and $p_{i,3}$ converges to $\lambda_1(A_d)$.

As mentioned previously, the structure of communication network is similar to the sparsity of matrix $A_d$ in order for each node to compute (5). Hence, since matrix $A_d$ is a discretization of original matrix $A$, the communication graph $\mathcal{G}$ will become fully connected. Next, it will be demonstrated that the proposed data-driven strategy can overcome this issue, i.e., it only requires a sparse and strongly connected digraph which is independent of the structure of matrix $A_d$.

### B. Data-driven power iteration

In the following, we develop a data-driven version of the algorithms presented in Section III-A. At a first glance, it seems that the knowledge of matrix $A_d$ is necessary to compute $z^{(1)}(k)$ from (5) for $k = 1, 2, \cdots$. However, by setting $\underline{z^{(1)}(0) = x(0)}$ and noting from dynamics (2) the relationship between the measurement $x(0), x(1), \cdots$, we can observe that $z^{(1)}(1)$ in (5) can be written as

$$z^{(1)}(1) = A_d z^{(1)}(0) = A_d x(0) = x(1).$$

Hence, the power iteration (5) can be embedded into the dynamics (2) by choosing properly the starting vector $z^{(1)}(0)$. In general, $z^{(1)}(k+1)$ can be written as

$$\boxed{z^{(1)}(k+1) = x(k+1).} \tag{6}$$

Given that data $x(k)$ is available, it can be seen from (6) that the matrix $A_d$ is not required to calculate $z^{(1)}(k)$. Therefore, the communication network topology $\mathcal{G}$ can be chosen to be any strongly connected digraphs so that each node can distributively execute steps 2–6 in Section III-A. Suppose that $x(0)$ satisfies $(x(0))^T \cdot w_1^{A_d} \neq 0$, the eigenvalue $\lambda_1(A_d)$ can be estimated in a distributed manner and the eigenvalue of interest $\lambda_1(A)$ can then be calculated from (3). Furthermore, it is known that for a large $k$, $z^{(1)}(k)$ under (5) or (6) will converge to $w_1^{A_d}$ when $\lambda_1(A_d) \in \mathbb{R}$ [14].

*Remark 3.1:* When iteration number $k$ is large, the state $x(k)$ may converge to zero which affects performance of the power iteration. To overcome this, a normalization step is added in practice. The proposed method can also be applied to power iteration which includes a normalization as in [7].

Next, let us consider the case where $\lambda_1(A_d) \in \mathbb{R}$ and we are interested in developing data-driven algorithm to distributively estimate $\lambda_2(A)$ after estimating $\lambda_1(A)$ and $w_1^{A_d}$. To this end, we consider the following update rule

$$z^{(2)}(k+1) = Q_1 z^{(2)}(k), \tag{7}$$

where matrix $Q_1$ is given by

$$Q_1 = A_d - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T \tag{8}$$

where $w_1^{A_d}$ denotes the right eigenvector corresponding to $\lambda_1(A_d)$. We then have the following lemma.

*Lemma 1:* For matrix $Q_1$ defined in (8), we have $\lambda_1(Q_1) = 0$ and $\lambda_i(Q_1) = \lambda_i(A_d)$ for $i = \{2, \cdots, n\}$.

*Proof:* It follows from $A_d w_1^{A_d} = \lambda_1(A_d) w_1^{A_d}$ and $\|w_1^{A_d}\| = 1$ that

$$Q_1 w_1^{A_d} = A_d w_1^{A_d} - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T w_1^{A_d} = 0$$

which implies that 0 is an eigenvalue of $Q_1$ and $w_1^{A_d}$ is the corresponding eigenvector. In addition, for $i = \{2, \cdots, n\}$, we can compute

$$\begin{aligned} Q_1^T \nu_i^{A_d} &= A_d^T \nu_i^{A_d} - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T \nu_i^{A_d} \\ &= \lambda_i(A_d) \nu_i^{A_d} - \lambda_1(A_d) w_1^{A_d} \underbrace{(w_1^{A_d})^T \nu_i^{A_d}}_{=0} = \lambda_i(A_d) \nu_i^{A_d} \end{aligned}$$

where $\nu_i^{A_d}$ denotes the left eigenvector of $A_d$ corresponding to $\lambda_i(A_d)$. Hence, for $i = \{2, \cdots, n\}$ eigenvalues $\lambda_i(A_d)$ are also the eigenvalues of $Q_1$ and $\nu_i^{A_d}$ are the corresponding left eigenvectors, which completes the proof. ■

Update rule (7) is a power iteration applied to deflated matrix $A_d - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T$. One of our contributions is that in contrast to most of model-based power iteration discussed in the literature such as the one presented in [7], [8], we propose to use the right eigenvector $w_1^{A_d}$ instead of the left eigenvector $\nu_1^{A_d}$ to define matrix $Q_1$. It will be shown later that this choice facilitates us and plays a key role in developing data-driven deflated power iteration.

It can be seen from Lemma 1 and (4) that $|\lambda_2(Q)| \geq |\lambda_3(Q)| \geq \cdots$ with $\lambda_i(Q) = \lambda_i(A_d)$ for $i = \{2, \cdots, n\}$. Each node can then estimate $\lambda_2(A_d)$ by executing steps 1–6 described in Section III-A and by substituting the iteration in (5) with (7). Next, similar to update rule (5) we will demonstrate how each node can compute $z^{(2)}(k)$ under (7) from sampled data $x(k)$. First, setting $\underline{z^{(2)}(0) = x(0)}$, we can write from (7) for $z^{(2)}(1)$ as

$$\begin{aligned} z^{(2)}(1) &= (A_d - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T) z^{(2)}(0) \\ &= x(1) - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T x(0). \end{aligned}$$

As can be observed, since each node has estimated $\lambda_1(A_d), w_1^{A_d}$ it can then compute $z^{(2)}(1)$ using sampled data $x(0), x(1)$. Similarly, by noting that $A_d w_1^{A_d} = \lambda_1(A_d) w_1^{A_d}$

we can write $z^{(2)}(2)$ as

$$\begin{aligned} z^{(2)}(2) &= (A_d - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T) z^{(2)}(1) \\ &= A_d x(1) - \lambda_1(A_d) A_d w_1^{A_d} (w_1^{A_d})^T x(0) \\ &\quad -\lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T x(1) \\ &\quad +\lambda_1^2(A_d) w_1^{A_d} (w_1^{A_d})^T w_1^{A_d} (w_1^{A_d})^T x(0) \\ &= x(2) - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T x(1). \end{aligned}$$

In general, we can write (7) as

$$\boxed{z^{(2)}(k+1) = x(k+1) - \lambda_1(A_d) w_1^{A_d} (w_1^{A_d})^T x(k).} \quad (9)$$

Hence, from (9) we can see that $z^{(2)}(k)$ under update rule (7) can be calculated using only available data $x(k)$ and $x(k-1)$. Note that for each node to compute $z_i^{(2)}(k) \in \mathbb{R}^{n_i}$ according to (9) in a distributed manner, it needs to be able to distributively calculate $(w_1^{A_d})^T x(k)$. To this end, observe that from discrete-time dynamics $x(k+1) = A_d x(k)$ we have $x(k) \to w_1^{A_d}$ for a large $k$ given that $x(0)^T w_1^{A_d} \neq 0$. Hence, for large value of $k$ node $i$ will know the vector

$$\overline{w}_i^{A_d} = [w^{A_d}_{1,(\sum_{r=1}^{i-1} n_r)+1}, \cdots, w^{A_d}_{1,\sum_{r=1}^{i} n_r}]^T$$

where $w_{1,i}^{A_d}$ denotes the $i$-th element of $w_1^{A_d}$. Next, we can write $(w_1^{A_d})^T x(k)$ as

$$(w_1^{A_d})^T x(k) = q\left(\frac{\sum_{i=1}^{q} (\overline{w}_i^{A_d})^T x_i(k)}{q}\right) = q x_{ave}(k). \quad (10)$$

Each node computes $x_{ave}(k)$ using the well-known finite-time average-consensus algorithm on strongly connected digraph $\mathcal{G}$, e.g., [15], [16] and by setting the initial value of its variable to $(\overline{w}_i^{A_d})^T x_i(k)$. Therefore, given that each node knows the total number of nodes $q$, the term $(w_1^{A_d})^T x(k)$ can be calculated in a distributed manner.

*Remark 3.2:* When the **eigenvalue** $\lambda_1(A_d)$ **is complex**, the iteration (5) or (6) will not converge to $w_1^{A_d}$. Hence, in this case it is not possible to apply the deflation technique to estimate $\lambda_2(A_d)$ and as a result the data-driven power iteration can only be used to estimate $\lambda_1(A_d)$.

Next, consider the case where $\lambda_1(A_d), \lambda_2(A_d) \in \mathbb{R}$. From previous discussions, the eigenvalues $\lambda_1(A_d), \lambda_2(A_d)$ and right eigenvector $w_1^{A_d}$ have already been estimated using available data. Furthermore, since $\lambda_2(A_d) = \lambda_2(Q_1) \in \mathbb{R}$ the update law (9) will converge to the right eigenvector corresponding to $\lambda_2(Q_1)$, that is $w_2^{Q_1}$, for a large $k$. By using the idea of deflation used to estimate $\lambda_2(A_d)$, we will show next that $\lambda_3(A_d)$ can also be estimated using available data. To this end, consider the following update rule

$$z^{(3)}(k+1) = Q_2 z^{(3)}(k) \quad (11)$$

where $z^{(3)}(k) \in \mathbb{R}^n$ and matrix $Q_2$ is given by

$$Q_2 = Q_1 - \lambda_2(Q_1) w_2^{Q_1} (w_2^{Q_1})^T. \quad (12)$$

Using similar argument as in the proof of Lemma 1, it can be concluded that $|\lambda_3(Q_2)| \geq |\lambda_4(Q_2)| \geq \cdots$, and $\lambda_i(Q_2) = \lambda_i(Q_1) = \lambda_i(A_d)$ for $i = \{3, 4, \cdots, n\}$. Therefore, if $z^{(3)}(k)$ can be calculated using available data, the eigenvalue $\lambda_3(A_d)$, respectively $\lambda_3(A)$, can be estimated in a distributed manner similar to $\lambda_1(A_d)$ and $\lambda_2(A_d)$. Following similar steps as the estimation of $\lambda_2(A_d)$, from (7), (11) and by setting initial value $\underline{z^{(3)}(0) = z^{(2)}(0)}$, we can write $z^{(3)}(1)$ as

$$\begin{aligned} z^{(3)}(1) &= [Q_1 - \lambda_2(Q_1) w_2^{Q_1} (w_2^{Q_1})^T] z^{(3)}(0) \\ &= z^{(2)}(1) - \lambda_2(Q_1) w_2^{Q_1} (w_2^{Q_1})^T z^{(2)}(0). \end{aligned}$$

In general, $z^{(3)}(k)$ can be expressed as

$$\boxed{z^{(3)}(k+1) = z^{(2)}(k+1) - \lambda_2(A_d) w_2^{Q_1} (w_2^{Q_1})^T z^{(2)}(k).} \quad (13)$$

Hence, It can be observed from (13) that in order to calculate $z^{(3)}(k+1)$, the subsystems only need to store the time-series values of vectors $z^{(2)}(k+1), z^{(2)}(k)$ calculated from (9). In general, for $j \geq 2$ and $\lambda_i(A) \in \mathbb{R}$ where $i = 1, \cdots, j-1$, we can write $z^{(j)}(k+1)$ as

$$\boxed{\begin{aligned} z^{(j)}(k+1) =& z^{(j-1)}(k+1) \\ & - \lambda_{j-1}(A_d) w_{j-1}^{Q_{j-2}} (w_{j-1}^{Q_{j-2}})^T z^{(j-1)}(k) \end{aligned}} \quad (14)$$

where $w_{j-1}^{Q_{j-2}}$ denotes the estimated right eigenvector corresponding to the eigenvalue $\lambda_{j-1}(A_d)$ and $\underline{z^{(j)}(0) = z^{(j-1)}(0)}$.

From the above analysis, we can conclude the following regarding data-driven distributed power iteration developed in this paper:

- The proposed distributed data-driven power iteration can be used to sequentially estimate dominant eigenvalues $\lambda_i(A)$ for $i = 1, \cdots$ until the rightmost complex eigenvalue (see remark 3.2)
- The data-driven distributed estimation of dominant eigenvalue $\lambda_i(A)$ only requires (needs to store) the information on $z^{(i-1)}(k)$ (or $x(k)$ when $i = 1$)
- Estimation of dominant eigenvalues only requires a single trajectory data $x(k)$ for $k = 0, 1, \cdots$
- The communication network topology can be chosen to be any strongly connected digraph.

## IV. Numerical Examples and Applications

In this section, we demonstrate and evaluate the proposed data-driven distributed power iteration using several numerical examples and applications. For the data-driven estimation, we set the sampling time $T = 0.03s$.

### A. Interconnected system with reducible physical structure

For the first example, we consider an interconnected system with $q = 3$ and $n_i = 1$ for all nodes $i$. Furthermore, matrix $A$ (which is unknown to the nodes) in (1) is given by

$$A = -\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

whose physical interconnection topology is shown in Fig. 2a. Note that the structure of its physical interconnection is not strongly connected. The eigenvalues of $A$ is given by $\lambda_i(A) = \{-1, -3, -4\}$. Each node then applies data-driven distributed power iteration proposed in the previous

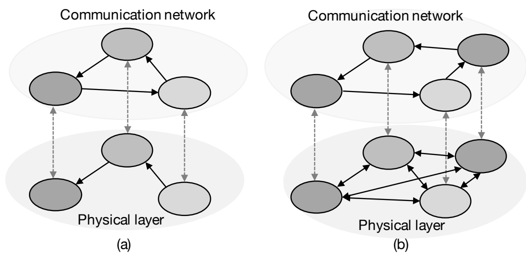


Fig. 2: Structures of physical interconnection and communication network used in the numerical examples

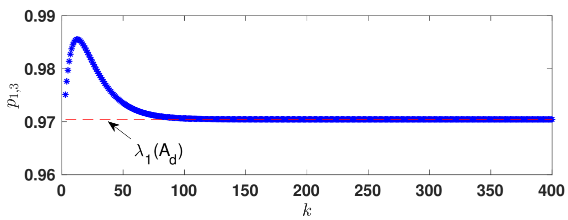


Fig. 3: Trajectory of $p_{1,3}$ computed by node 1 for the example in Section IV-A

section with $T = 0.03$ and whose communication network topology is shown in Fig. 2a to estimate $\lambda_1(A) = -1$ or $\lambda_1(A_d) = 0.9704$. As can be observed from Fig. 3, the value of $p_{1,3}(k)$ converges to $\lambda_1(A_d)$. The other eigenvalues can also then be estimated from (9), (13).

### *B. Interconnected system with fully connected subsystems*

For the second example, we consider an interconnected system consisting of four subsystems where $q = 4$ and $n_i = 2$ for all subsystems. Furthermore, the unknown matrix $A$ is given by

$$A = \begin{bmatrix} 0 & 3.14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.9 & -0.9 & 0.42 & 0 & 0.33 & 0 & 0.15 & 0 \\ 0 & 0 & 0 & 3.14 & 0 & 0 & 0 & 0 \\ 0.36 & 0 & -0.9 & -0.5 & 0.3 & 0 & 0.24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.14 & 0 & 0 \\ 0.15 & 0 & 0.15 & 0 & -0.6 & -0.8 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.14 \\ 0.3 & 0 & 0.39 & 0 & 0.51 & 0 & -1.2 & -0.4 \end{bmatrix}$$

whose physical interconnection topology is shown in Fig. 2b. The rightmost eigenvalues of $A$ are equal to $\lambda_1(A) = 0, \lambda_2(A) = -0.2215 + 2.2119i, \lambda_3(A) = -0.2215 - 2.1219i$ or $\lambda_1(A_d) = 1, \lambda_2(A_d) = 0.9914 + 0.0632i, \lambda_3(A_d) = 0.9914 - 0.0632i$. The nodes apply data-driven distributed power iteration in the previous section to distributively estimate $\lambda_2(A)$ (i.e., $\lambda_2(A_d)$) after estimating $w_1^{A_d}$ whose communication network topology is depicted in Fig. 2b. Note that the communication topology is sparse even though the physical interconnection structure is fully connected. As can be observed from Fig. 4, the value of $p_{3,1}(k)$ (computed by node 3) converges to $\lambda_2(A_d)$ and $\lambda_2(A)$ can be computed from (3). Since $\lambda_2(A)$ is a complex eigenvalue, the data-driven power iteration cannot be utilized to estimate $\lambda_4(A)$.

### *C. Estimation of generalized algebraic connectivity*

Let us consider the following consensus dynamics

$$\dot{x} = -Lx = Ax$$

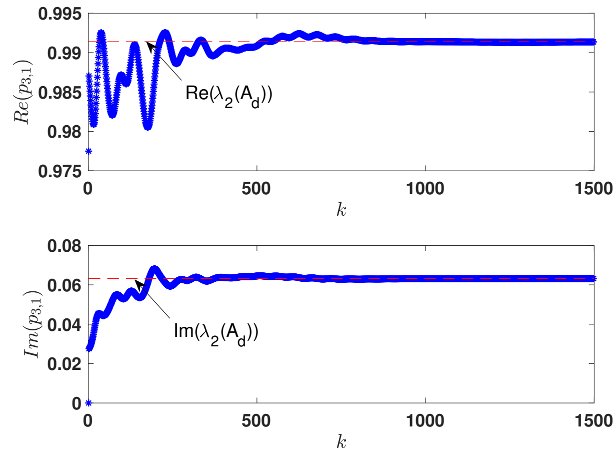


Fig. 4: Trajectories of $p_{3,1}$ computed by node 3 for the example in Section IV-B

where matrix $L$ is a weighted Laplacian matrix whose structure corresponds to a strongly connected digraph [2]. The consensus dynamics has many applications such as in robotic network [2] and intelligent transportation system [17]. Due to property of the Laplacian matrix, it is known that $\lambda_1(L) = 0$ with corresponding right eigenvector $w_1^L = \mathbf{1}$ and $\Re(\lambda_i(L)) > 0$ for $i \neq 1$. Generalized algebraic connectivity defined by

$$\tilde{\lambda}(L) = \min_{\lambda_i(L) \neq 0} \Re(\lambda_i(L))$$

measures network connectivity and convergence rate for reaching consensus [3].

A centralized algorithm to estimate $\tilde{\lambda}(L)$ is presented in [3] given that matrix $L$ is known. If the data $x(k)$ is available, the proposed data-driven power iteration algorithm can be used to distributively estimate generalized algebraic connectivity $\tilde{\lambda}(L)$ without requiring knowledge of matrix $L$. To this end, it can be observed that $\lambda_2(A) = -\tilde{\lambda}(L)$. Since we know that $\lambda_1(A) = 0$ and $w_1^L = \mathbf{1}$, each node can then implement data-driven distributed power iteration (9) to estimate $\lambda_2(A_d)$ and the eigenvalue $\lambda_2(A)$ can be estimated from (3).

To demonstrate the idea, we consider a network of six nodes and consensus dynamics similar to the one used in [3] where matrix $A$ is given by

$$A = \begin{bmatrix} -2.9 & 0.1 & 0.3 & 0.8 & 0.9 & 0.8 \\ 0.2 & -1.9 & 1 & 0.1 & 0.5 & 0.1 \\ 0.1 & 0.9 & -2.7 & 0.7 & 0.9 & 0.1 \\ 1 & 0.5 & 0.6 & -3.1 & 0.4 & 0.6 \\ 0 & 0.2 & 0.3 & 0.1 & -1.6 & 1 \\ 0.5 & 0.4 & 0.6 & 0.1 & 0.1 & -1.7 \end{bmatrix}.$$

Each node executes data-driven power iteration (9) whose communication network topology is similar to existing communication structure between the nodes (i.e., the structure of matrix $A$). An example of estimation at node 5 is shown in Fig. 5. As can be seen, node 5 could accurately estimate $\lambda_2(A_d)$. From (3), the generalized algebraic connectivity $\tilde{\lambda}(L) = -\lambda_2(A)$ is equal to $\tilde{\lambda}(L) = 2.1951 \pm 0.4312i$ (note that the actual value is $\tilde{\lambda}(L) = 2.1916 \pm 0.4313i$).

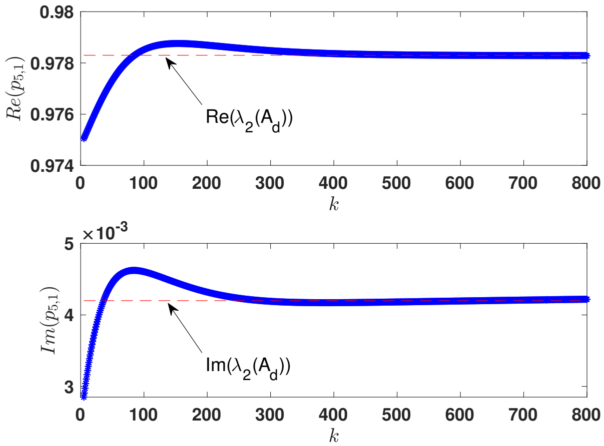


Fig. 5: Trajectories of $p_{5,1}$ computed by node 5 for the example in Section IV-C

### *D. Estimation of dominant eigenvector for SIS epidemic model*

Finally, consider a network of $q$ nodes and the susceptible-infectious-susceptible (SIS) epidemic model [18] given by

$$x(k+1) = [(1-\delta)I + \beta P]x(k) = A_d x(k) \tag{15}$$

where $P$ denotes the adjacency matrix corresponding to the network which is assumed to be undirected and connected, parameter $\delta > 0$ is the curing rate on an infected node and $\beta > 0$ is the infection rate on a link connected to an infected node. Furthermore, the state $x_i(k)$ denotes probability that node $i$ is infected at time-step $k$. It is known that the largest eigenvalue of a symmetric adjacency matrix $P$ denoted by $\lambda_{max}(P)$ plays an important role in the dissemination of disease in a network [18]. In particular, we are interested in estimating the eigenvector corresponding to $\lambda_{max}(P)$, i.e., $w^P_{max}$. The right eigenvector $w^P_{max}$ can be used to develop a distributed strategy for removing a fraction of links from a network in order to slow down the spread of disease in the network, see the discussions in [19] for the details.

The proposed data-driven power iteration can be utilized to distributively estimate $w^P_{max}$ from available data $x(k)$. To this end, it can be seen from (15) that $w^P_{max} = w^{A_d}_1$ [20]. Furthermore, since matrix $A_d$ is primitive, the spectral radius of $A_d$ is equal to $\lambda_1(A_d)$ [19]. Hence, each node can execute data-driven power iteration in (6) without requiring the knowledge of parameters $\beta, \delta$ and will converge to $w^{A_d}_1$, i.e., the eigenvector of interest $w^P_{max}$.

## V. Conclusion & Future work

We propose data-driven power iteration to estimate the dominant eigenvalues of an unknown linear time-invariant system in a distributed manner using a single trajectory data. In particular, the proposed data-driven algorithm estimates sequentially the dominant eigenvalues until the rightmost complex one. In order to perform the distributed estimation, the communication network topology can be chosen to be any strongly connected directed graphs. This provides an additional degree-of-freedom for the designer in choosing or optimizing the communication network topology independent from the physical interconnection topology. Future work include analysis of the proposed algorithm in the presence of noisy measurements and development of finite-time algorithm to deal with limited available data.

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